

Central Limit Theorem for Coloured Hard-Dimers

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Abstract

Using an averaged generating function for coloured hard-dimers, some random variables of interest are studied. The main result lies in the fact that all their probability distributions obey a central limit theorem.

1 Introduction

In the literature coloured hard-dimers are applied in the framework of causally triangulated $(2 + 1)$ -dimensional quantum gravity. It was proved in [1], by using special triangulations of spacetime, that the generating function of the one step propagator depends on that one of coloured hard-dimers. For any configuration ξ_N , of length N , of blue and red sites, a coloured hard-dimer is a sequence of blue and red dimers, that satisfy the “hardness” condition, i.e. they can not intersect. A dimer is an edge connecting two nearest sites of the same colour.

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In the present paper we consider the “averaged” generating function for coloured hard-dimers, that is, the mean of generating functions over all the configurations ξ_N , with N fixed. In [2] we have found an explicit formula for it together with estimates from above and from below, that are both exponential, for large N . In the following we study the probability distribution associated with the averaged generating function. Then we analyze the probability distributions corresponding to some random variable (r.v.) of interest. In particular the number of dimers and the total length of them. It turns out that the role played by the r.v. that measures the total number of dimers and single points (i.e. sites not occupied by dimers) is very important. We prove that the r.v. total length of dimers is binomial with parameters $N - 1$ and $\frac{1}{3}$. Moreover we see that, even though the number of dimers has an unknown probability distribution, we are able to estimate its mean and variance asymptotically, by using some recursive formulas that relate its moments with that ones resulting from the dimers’ length. Although the dimers’ number distribution is not binomial, its variance is of order N , as $N \rightarrow \infty$, as in the binomial case.

The main result of the present article is a local Central Limit Theorem (C.L.T.), for N large enough, for the joint probability distribution corresponding to the number of dimers and that one of dimers and single points. The limit distribution is a bivariate gaussian distribution with correlation coefficient equal to $-\frac{1}{\sqrt{3}}$. Hence a C.L.T. holds, as $N \rightarrow \infty$, also for the marginal probability distribution related to the dimers’ number.

The paper is organized as follows. In section 2 we define the probability distribution associated with coloured hard-dimers, through the averaged generating function, and find an exact expression for its normalizing constant $C_N = (\frac{3}{2})^{N-1}$. Moreover we recognize the right probability distribution for the length of dimers. In section 3 we calculate the first two moments of the dimers’ number. Finally in section 4 we prove a C.L.T. for the dimers’ number.

2 Coloured hard-dimers and probability distributions

Given a sequence ξ_N of length N of blue and red sites on the one-dimensional lattice \mathbb{Z} , one defines a dimer to be an edge connecting two nearest sites of the same colour, that characterizes the dimer colour. A sequence of coloured and non overlapping dimers in turn yields a “coloured hard-dimer”. In Fig.1

an example of a coloured hard-dimer is given.



Figure 1: A hard-dimer, $N = 13$

As described in [1] and [2] one introduces the generating function associated to coloured hard-dimers on ξ_N

$$Z_{\xi_N}(u, v, w) = \sum_D u^{n_b(D)} v^{n_r(D)} w^{n_{br}(D)}, \quad u, v, w \in (0, \infty),$$

where D is a hard-dimer on ξ_N , $n_b(D)$ and $n_r(D)$ indicate the number of blue and red dimers respectively on D , and $n_{br}(D)$ the total number of sites within each dimer (this means the sites having a colour different from the colour of dimers containing them). Moreover, let us define by $\gamma_b(D)$ and $\gamma_r(D)$ the number of blue and red sites of D respectively, not occupied by dimers (“single points”). Then the following constraint

$$2n_b(D) + 2n_r(D) + n_{br}(D) + \gamma_b(D) + \gamma_r(D) = N \quad (2.1)$$

holds. In the above example $n_b(D) = 1$, $n_r(D) = 2$, $n_{br}(D) = 3$, $\gamma_b(D) = 3$ and $\gamma_r(D) = 1$.

In [2] we studied the average of Z_{ξ_N} over all the configurations ξ_N , i.e.

$$\tilde{Z}_N(u, v, w) = \frac{1}{2^N} \sum_{\xi_N} Z_{\xi_N}(u, v, w) \quad (2.2)$$

and found estimates from above and from below for \tilde{Z}_N . In order to obtain them we proved an explicit formula for the mean \tilde{Z}_N , by using combinatorial tools

$$\tilde{Z}_N(u, v, w) = 1 + \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \binom{N-t+s}{s} \binom{t-s-1}{s-1} \left(\frac{u+v}{4} \right)^s \left(\frac{w}{2} \right)^{t-2s} \quad (2.3)$$

($\lfloor \cdot \rfloor$ denotes the integer part).

Now, let us define a family of probability spaces $(\Omega_N, \mathcal{F}_N, P_N)_{N \in \mathbb{N}_{\geq 1}}$. We choose Ω_N to be the set of all different hard-dimer configurations, where

by hard-dimer configuration we mean a particular sequence ξ_N of length N together with a particular hard-dimer on ξ_N . The σ -algebra \mathcal{F}_N is the set of all subsets of Ω_N and for P_N we take the probability measure which is distributed uniformly on Ω_N . Normalizing the function \tilde{Z}_N by $\tilde{Z}_N|_{(u,v,w)=(1,1,1)}$ then just gives the joint generating function of the random variables n_b, n_r, n_{br} , defined on $(\Omega_N, \mathcal{F}_N, P_N)$, which count, for each hard-dimer configuration, the number of blue, red dimers and the total number of sites within dimers, respectively.

The random variable $n_b + n_r$ (characterized by the index s in formula (2.3)) gives the number of dimers for each hard-dimer configuration. Correspondingly, the random variable $N - \gamma_b - \gamma_r$ (indexed by t in (2.3)) denotes the number of sites occupied by dimers. The above formula (2.3) is obtained by fixing first the number of blue dimers (n_b), red ones (n_r) and single points (γ_b and γ_r), i.e., $n_b + n_r + \gamma_b + \gamma_r$, indexed by $N - t + s$, without assigning any length to dimers, even though the total length of dimers, defined as $n_b + n_r + n_{br}$, (indexed by $t - s$) is given because of (2.1). This gives a factor

$$\frac{(n_b + n_r + \gamma_b + \gamma_r)!}{n_b! n_r! \gamma_b! \gamma_r!}$$

Then, in the non-trivial case where $n_b + n_r \neq 0$, one can assign a length to each dimer, taking into account that the total length of them is given by $n_b + n_r + n_{br}$, contributing another factor $\binom{n_b + n_r + n_{br} - 1}{n_b + n_r - 1}$. By summing over γ_b, n_b and then over s and t , we get the expression (2.3). See [2] for the details.

In the present paper we want to analyze the random variables $n_b + n_r$ and $n_b + n_r + n_{br}$, more precisely their probability distribution. We prove that the random variable $n_b + n_r + n_{br}$ is binomial with parameters $N - 1$ and $\frac{1}{3}$ and, hence, a Central Limit Theorem (C.L.T.) holds, for large N (De Moivre-Laplace's Theorem). In the case of $n_b + n_r$ we see that the probability distribution is unknown, but, fixing the random variable $n_b + n_r + \gamma_b + \gamma_r$, which in turn is binomial with parameters $N - 1$ and $\frac{2}{3}$, because of (2.1), the conditional distribution of $n_b + n_r$ is hypergeometric. Moreover, for N large enough, a C.L.T. holds for the joint probability distribution of $n_b + n_r$ and $n_b + n_r + \gamma_b + \gamma_r$ and, hence, also for the distribution of $n_b + n_r$. For the proof of these results the random variable $n_b + n_r + \gamma_b + \gamma_r$ plays a very important part.

Evaluating the averaged generating function \tilde{Z}_N at the point $u = v = w = 1$ we derive the normalizing constant of the probability measure P_N associated to coloured hard-dimers

$$C_N = \tilde{Z}_N|_{(u,v,w)=(1,1,1)} = 1 + \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \binom{N-t+s}{s} \binom{t-s-1}{s-1} \frac{1}{2^{t-s}} \quad (2.4)$$

This also shows that the joint probability distribution \tilde{P}_N related to the r.v.s $n_b + n_r$ and $2(n_b + n_r) + n_{br}$, more precisely

$$\tilde{P}_N(s, t) = P_N(n_b + n_r = s; 2(n_b + n_r) + n_{br} = t) \quad (2.5)$$

is given by

$$\tilde{P}_N(s, t) = \begin{cases} \frac{1}{C_N} & \text{for } s = t = 0 \\ \frac{1}{C_N} \binom{N-t+s}{s} \binom{t-s-1}{s-1} \frac{1}{2^{t-s}} & \text{otherwise} \end{cases} \quad (2.6)$$

The main result of this section is an explicit formula for the normalizing constant C_N , that holds for any N . It is obtained by using combinatoric arguments.

Theorem 2.1 *For any N , the following formula*

$$C_N = \left(\frac{3}{2}\right)^{N-1} \quad (2.7)$$

holds.

Proof: Consider the following change of variables $t' = N - t, k = N - t + s$, so that (2.4) becomes (note that after changing the variables, we shall rename t' again by t)

$$C_N = 1 + \left(\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \sum_{t=0}^{k-1} + \sum_{k=\lfloor \frac{N}{2} \rfloor + 1}^{N-1} \sum_{t=2k-N}^{k-1} \right) \binom{k}{t} \binom{N-k-1}{k-t-1} \frac{1}{2^{N-k}} \quad (2.8)$$

Note that in formula (2.8) only the combinatorial coefficients depend on t . Moreover, the combinatorial coefficients of the first sum in (2.8) $\binom{k}{t} \binom{N-k-1}{k-t-1}$ yield a non-normalized hypergeometric distribution with parameters $N - 1$ (population size), k (number of successes in the population) and $k - 1$ (sample size). Therefore summing over t we get

$$\sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2^{N-k}} \sum_{t=0}^{k-1} \binom{k}{t} \binom{N-k-1}{k-t-1} = \sum_{k=1}^{\lfloor \frac{N}{2} \rfloor} \frac{1}{2^{N-k}} \binom{N-1}{k-1} \quad (2.9)$$

Analogously, performing the change of variable $t' = t - 2k + N$, we get in the second sum of (2.8) again a non-normalized hypergeometric distribution with parameters $N - 1, N - k - 1, N - k - 1$. Summing again over t we get

$$\begin{aligned}
& \sum_{k=\lfloor \frac{N}{2} \rfloor + 1}^{N-1} \frac{1}{2^{N-k}} \sum_{t=2k-N}^{k-1} \binom{k}{t} \binom{N-k-1}{k-t-1} = \\
& \sum_{k=\lfloor \frac{N}{2} \rfloor + 1}^{N-1} \frac{1}{2^{N-k}} \sum_{t=0}^{N-k-1} \binom{k}{N-k-t} \binom{N-k-1}{t} = \\
& \sum_{k=\lfloor \frac{N}{2} \rfloor + 1}^{N-1} \frac{1}{2^{N-k}} \binom{N-1}{k-1} \tag{2.10}
\end{aligned}$$

with the convention that the binomial coefficient $\binom{n}{k} = 0$ if $k > n$. Putting together the last terms in (2.9) and (2.10) we get a binomial formula

$$\sum_{k=1}^{N-1} \frac{1}{2^{N-k}} \binom{N-1}{k-1} = \frac{1}{2^{N-1}} \sum_{k=0}^{N-1} \binom{N-1}{k} 2^k - 1 = \left(\frac{3}{2}\right)^{N-1} - 1 \tag{2.11}$$

□

An important Corollary of the previous Theorem is that the probability distribution of the r.v. $n_b + n_r + \gamma_b + \gamma_r$ (total number of dimers and single points) related to the index k in (2.8) has a binomial distribution with parameters $N - 1$ and $\frac{2}{3}$.

Corollary 2.1 *The probability distribution of the r.v. $n_b + n_r + \gamma_b + \gamma_r$ is a binomial distribution with parameters $N - 1$ and $\frac{2}{3}$.*

Proof: Performing the same change of variables as for (2.8) and summing the probability distribution with respect to t , as in (2.8) we get

$$\begin{aligned}
P_N(n_b + n_r + \gamma_b + \gamma_r = k) &= \left(\frac{2}{3}\right)^{N-1} \binom{N-1}{k-1} \frac{1}{2^{N-k}} = \\
& \binom{N-1}{k-1} \left(\frac{2}{3}\right)^{k-1} \left(\frac{1}{3}\right)^{N-k}
\end{aligned}$$

for $1 \leq k \leq N - 1$. For $k = N$ it holds $n_b + n_r = 0 \Rightarrow \gamma_b + \gamma_r = N$. Hence

$$P_N(n_b + n_r + \gamma_b + \gamma_r = N) = P_N(n_b + n_r = 0) = \frac{1}{C_N} = \left(\frac{2}{3}\right)^{N-1}$$

by (2.6). □

Corollary 2.2 *For large N a C.L.T. (De Moivre-Laplace) for the r.v. $n_b + n_r + \gamma_b + \gamma_r$ holds*

$$P_N(n_b + n_r + \gamma_b + \gamma_r = k) = \frac{1}{\sqrt{\frac{4}{9}\pi(N-1)}} e^{-\frac{(k - \frac{2}{3}(N-1))^2}{\frac{4}{9}(N-1)}} (1 + r_k(N))$$

where $\lim_{N \rightarrow \infty} r_k(N) = 0$, uniformly with respect to k , for any k such that $y = [k - \frac{2}{3}(N-1)] / \sqrt{\frac{2}{9}(N-1)}$ belongs to a finite interval $(-A, A)$.

Analogously one can find a similar result for the r.v. $n_b + n_r + n_{br}$ (total length of dimers).

Corollary 2.3 *The probability distribution of the r.v. $n_b + n_r + n_{br}$ is a binomial distribution with parameters $N-1$ and $\frac{1}{3}$.*

Proof: Taking into account (2.1) one has

$$P_N(n_b + n_r + n_{br} = h) = P_N(n_b + n_r + \gamma_b + \gamma_r = N - h) = \binom{N-1}{h} \left(\frac{1}{3}\right)^h \left(\frac{2}{3}\right)^{N-1-h}$$

for $0 \leq h \leq N-1$. □

Corollary 2.4 *For large N a C.L.T. (De Moivre-Laplace) for the r.v. $n_b + n_r + n_{br}$ holds*

$$P_N(n_b + n_r + n_{br} = h) = \frac{1}{\sqrt{\frac{4}{9}\pi(N-1)}} e^{-\frac{(h - \frac{1}{3}(N-1))^2}{\frac{4}{9}(N-1)}} (1 + r'_h(N))$$

where $\lim_{N \rightarrow \infty} r'_h(N) = 0$, uniformly with respect to h , for any h such that $z = [h - \frac{1}{3}(N-1)] / \sqrt{\frac{2}{9}(N-1)}$ belongs to a finite interval $(-B, B)$.

3 Number of dimers: moments

In the present section we investigate the distribution of the random variable $n_b + n_r$, more precisely, we calculate the first two moments of it, by using recursive asymptotic formulas that depend on the first two moments of the binomial r.v. $n_b + n_r + n_{br}$, studied in the previous section. Starting from

the averaged generating function \tilde{Z}_N , defined in (2.3), we rescale it by the normalizing constant C_N , calculated in the previous section

$$Z_N(u, v, w) = \frac{\tilde{Z}_N(u, v, w)}{C_N} = \left(\frac{2}{3}\right)^{N-1} \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \binom{N-t+s}{s} \binom{t-s-1}{s-1} \frac{1}{2^{t-s}} \left(\frac{u+v}{2}\right)^s w^{t-2s} \quad (3.1)$$

Therefore we calculate the moments of $n_b + n_r$ through its derivatives (see [3]). By symmetry of the variables u and v in (3.1) we have

$$\begin{aligned} E_N(n_b) &= \frac{\partial}{\partial u} Z_N(u, v, w)|_{(u,v,w)=(1,1,1)} = \\ \frac{\partial}{\partial v} Z_N(u, v, w)|_{(u,v,w)=(1,1,1)} &= E_N(n_r) = \\ \left(\frac{2}{3}\right)^{N-1} \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \frac{s/2}{2^{t-s}} \binom{N-t+s}{s} \binom{t-s-1}{s-1} \end{aligned}$$

We indicate by E_N the mean of random variables with respect to the probability measure P_N . Therefore

$$\begin{aligned} E_N(n_b + n_r) &= \left(\frac{2}{3}\right)^{N-1} \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \frac{s}{2^{t-s}} \binom{N-t+s}{s} \binom{t-s-1}{s-1} = \\ &= \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} s \tilde{P}_N(s, t) \end{aligned} \quad (3.2)$$

We could deduce the above formula (3.2) by noting that the index related to the r.v. $n_b + n_r$ is s . Analogously we find the formula for the mean of the r.v. $n_b + n_r + n_{br}$

$$E_N(n_b + n_r + n_{br}) = E_N(n_b + n_r) + E_N(n_{br})$$

and

$$\begin{aligned} E_N(n_{br}) &= \frac{\partial}{\partial w} Z_N(u, v, w)|_{(u,v,w)=(1,1,1)} = \\ \left(\frac{2}{3}\right)^{N-1} \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \frac{t-2s}{2^{t-s}} \binom{N-t+s}{s} \binom{t-s-1}{s-1} &= \end{aligned}$$

$$\sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} (t - 2s) \tilde{P}_N(s, t)$$

Hence

$$E_N(n_b + n_r + n_{br}) = \left(\frac{2}{3}\right)^{N-1} \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \frac{t-s}{2^{t-s}} \binom{N-t+s}{s} \binom{t-s-1}{s-1} \quad (3.3)$$

In the next proposition we prove a recursive asymptotic formula, that links the mean of the r.v. $n_b + n_r$ to that one of $n_b + n_r + n_{br}$, for N large enough.

Proposition 3.1 *The following asymptotics*

$$E_N(n_b + n_r) \asymp \frac{N}{3} - E_N(n_b + n_r + n_{br}) + \frac{2}{3} E_{N-1}(n_b + n_r + n_{br}) \quad (3.4)$$

holds, for N large enough.

Remark 3.1 *From Corollary 2.3 we have that under P_N , $n_b + n_r + n_{br} \sim B(N-1, \frac{1}{3})$ (binomial distribution), so that under P_{N-1} , $n_b + n_r + n_{br} \sim B(N-2, \frac{1}{3})$. Hence*

$$\begin{aligned} E_N(n_b + n_r + n_{br}) &= \frac{N-1}{3} \\ E_{N-1}(n_b + n_r + n_{br}) &= \frac{N-2}{3} \end{aligned} \quad (3.5)$$

From Remark 3.1 the next Corollary follows:

Corollary 3.1 *For large N ,*

$$E_N(n_b + n_r) \asymp \frac{2N-1}{9} \quad (3.6)$$

Proof: Applying the formula (3.4) and taking into account (3.5) we obtain

$$E_N(n_b + n_r) \asymp \frac{N}{3} - \frac{N-1}{3} + \frac{2}{3} \frac{N-2}{3} = \frac{2N-1}{9}$$

□

Remark 3.2 By identity (2.1) and from (3.5) and (3.6) we are able to calculate asymptotically the single point number's mean. In fact

$$E_N(\gamma_b + \gamma_r) = N - E_N(n_b + n_r) - E_N(n_b + n_r + n_{br}) \asymp$$

$$N - \frac{2N-1}{9} - \frac{N-1}{3} = \frac{4}{9}(N+1)$$

Note that if we consider only the first order of the asymptotics with respect to N , we have

$$E_N(\gamma_b + \gamma_r) \asymp \frac{4}{9}N \asymp 2E_N(n_b + n_r)$$

that is, for the present model the expected number of single points is asymptotically twice the expected number of dimers. Moreover, fixing the number of single points, the conditional probability distribution of γ_b (γ_r) is binomial and symmetric.

Proof of Proposition 3.1: From (3.2) we have

$$E_N(n_b + n_r) = \left(\frac{2}{3}\right)^{N-1} \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \frac{N-t+s}{2^{t-s}} \binom{N-t+s-1}{s-1} \binom{t-s-1}{s-1} \quad (3.7)$$

We applied in (3.7) the identity

$$s \binom{N-t+s}{s} = (N-t+s) \binom{N-t+s-1}{s-1}$$

Moreover, by Pascal's identity

$$\binom{N-t+s-1}{s-1} = \binom{N-t+s}{s} - \binom{N-t+s-1}{s}$$

the mean $E_N(n_b + n_r)$ becomes

$$\left(\frac{2}{3}\right)^{N-1} \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \frac{N-t+s}{2^{t-s}} \left[\binom{N-t+s}{s} - \binom{N-1-t+s}{s} \right] \binom{t-s-1}{s-1} \quad (3.8)$$

Taking into account the definition of the probability distribution \tilde{P}_N in (2.6), the formula (3.8) becomes

$$E_N(n_b + n_r) = \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} (N-t+s) \tilde{P}_N(s, t) - \frac{2}{3} \sum_{t=1}^{N-1} \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} (N-t+s) \tilde{P}_{N-1}(s, t) \quad (3.9)$$

In fact the normalizing constant for \tilde{P}_{N-1} is $C_{N-1} = \left(\frac{3}{2}\right)^{N-2}$, instead of C_N . The first sum in (3.9) gives

$$\begin{aligned} N \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \tilde{P}_N(s, t) - \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} (t-s) \tilde{P}_N(s, t) = \\ N \left[1 - \left(\frac{2}{3} \right)^{N-1} \right] - E_N(n_b + n_r + n_{br}) \asymp \\ N - E_N(n_b + n_r + n_{br}) \end{aligned} \quad (3.10)$$

by (2.6).

Analogously for the second sum in (3.9) one has

$$\begin{aligned} -\frac{2}{3} \left[N \sum_{t=1}^{N-1} \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \tilde{P}_{N-1}(s, t) - \sum_{t=1}^{N-1} \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} (t-s) \tilde{P}_{N-1}(s, t) \right] = \\ -\frac{2}{3} N \left[1 - \left(\frac{2}{3} \right)^{N-2} \right] + \frac{2}{3} E_{N-1}(n_b + n_r + n_{br}) \asymp \\ -\frac{2}{3} N + \frac{2}{3} E_{N-1}(n_b + n_r + n_{br}) \end{aligned} \quad (3.11)$$

Both formulas (3.10) and (3.11) give (3.4) and Proposition 3.1 is so proved. \square

In order to find the variance of $n_b + n_r$ we need an analogous recursive formula for the second factorial moment of $n_b + n_r$ and that one of $n_b + n_r + n_{br}$, whose distribution is well known. By (3.2) the second factorial moment of $n_b + n_r + n_{br}$ is of the form

$$\begin{aligned} E_N[(n_b + n_r)(n_b + n_r - 1)] = \\ \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} s(s-1) \tilde{P}_N(s, t) \end{aligned} \quad (3.12)$$

Analogously it easy to see that the second factorial moment of $n_b + n_r + n_{br}$ is

$$\begin{aligned} E_N[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] = \\ \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} (t-s)(t-s-1) \tilde{P}_N(s, t) \end{aligned} \quad (3.13)$$

In fact, the indices related to the r.v.s $n_b + n_r$ and $n_b + n_r + n_{br}$ are s and $t - s$ respectively.

We generalize the asymptotic recursive formula for the first moments, given in Proposition 3.1 to the second factorial moment of $n_b + n_r$ in terms of the first two factorial moments of $n_b + n_r + n_{br}$.

Proposition 3.2 *For N large enough, the following asymptotic recursive formula*

$$\begin{aligned} E_N[(n_b + n_r)(n_b + n_r - 1)] &\asymp \frac{N(N-1)}{9} + \\ &-2(N-1)[E_N(n_b + n_r + n_{br}) - \frac{4}{3} E_{N-1}(n_b + n_r + n_{br}) + \frac{4}{9} E_{N-2}(n_b + n_r + n_{br})] + \\ &E_N[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] - \frac{4}{3} E_{N-1}[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] + \\ &\frac{4}{9} E_{N-2}[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] \end{aligned} \quad (3.14)$$

holds.

Proof of Proposition 3.2: From (3.12) and taking into account the proof of Proposition 3.1, we have

$$\begin{aligned} E_N[(n_b + n_r)(n_b + n_r - 1)] &= \\ &\left(\frac{2}{3}\right)^{N-1} \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \frac{(N-t+s)(N-t+s-1)}{2^{t-s}} \binom{N-t+s-2}{s-2} \binom{t-s-1}{s-1} \end{aligned}$$

because of the identity

$$s(s-1) \binom{N-t+s}{s} = (N-t+s)(N-t+s-1) \binom{N-t+s-2}{s-2}$$

Considering the expressions (3.12) and (3.13) of the second factorial moments of $n_b + n_r$ and $n_b + n_r + n_{br}$ respectively, we rewrite $(N-t+s)(N-t+s-1)$ in a suitable form

$$(N-t+s)(N-t+s-1) = N(N-1) - 2(N-1)(t-s) + (t-s)(t-s-1) \quad (3.15)$$

Moreover, as in Proposition 3.1, we apply (this time twice) Pascal's formula

$$\binom{N-t+s-2}{s-2} = \binom{N-t+s-1}{s-1} - \binom{N-t+s-2}{s-1} =$$

$$\binom{N-t+s}{s} - 2\binom{N-t+s-1}{s} + \binom{N-t+s-2}{s} \quad (3.16)$$

By (3.15) and (3.16) the second factorial moment of $n_b + n_r$ becomes

$$\begin{aligned} E_N[(n_b + n_r)(n_b + n_r - 1)] = & \left(\frac{2}{3}\right)^{N-1} \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} [N(N-1) - 2(N-1)(t-s) + (t-s)(t-s-1)] \times \\ & \left[\binom{N-t+s}{s} - 2\binom{N-t+s-1}{s} + \binom{N-t+s-2}{s} \right] \binom{t-s-1}{s-1} \frac{1}{2^{t-s}} = \\ & \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} [N(N-1) - 2(N-1)(t-s) + (t-s)(t-s-1)] \tilde{P}_N(s, t) + \\ & - \frac{4}{3} \sum_{t=1}^{N-1} \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} [N(N-1) - 2(N-1)(t-s) + (t-s)(t-s-1)] \tilde{P}_{N-1}(s, t) + \\ & + \frac{4}{9} \sum_{t=1}^{N-2} \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} [N(N-1) - 2(N-1)(t-s) + (t-s)(t-s-1)] \tilde{P}_{N-2}(s, t) \quad (3.17) \end{aligned}$$

where the normalizing constant for \tilde{P}_{N-2} is $C_{N-2} = \left(\frac{3}{2}\right)^{N-3}$.

The first sum of the right hand side of (3.17) becomes

$$\begin{aligned} & N(N-1) \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} \tilde{P}_N(s, t) - 2(N-1) \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} (t-s) \tilde{P}_N(s, t) + \\ & \sum_{t=1}^N \sum_{s=1}^{\lfloor \frac{t}{2} \rfloor} (t-s)(t-s-1) \tilde{P}_N(s, t) = \\ & N(N-1) \left[1 - \left(\frac{2}{3}\right)^{N-1} \right] - 2(N-1) E_N(n_b + n_r + n_{br}) + \\ & E_N[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] \asymp N(N-1) - 2(N-1) E_N(n_b + n_r + n_{br}) + \\ & E_N[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] \quad (3.18) \end{aligned}$$

As above the second sum in (3.17) is of the form

$$-\frac{4}{3} N(N-1) \left[1 - \left(\frac{2}{3}\right)^{N-2} \right] + \frac{8}{3} (N-1) E_{N-1}(n_b + n_r + n_{br}) +$$

$$\begin{aligned}
& -\frac{4}{3}E_{N-1}[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] \asymp \\
& -\frac{4}{3}N(N-1) + \frac{8}{3}(N-1)E_{N-1}(n_b + n_r + n_{br}) + \\
& -\frac{4}{3}E_{N-1}[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] \tag{3.19}
\end{aligned}$$

Finally the last sum in (3.17) gives the following contribution

$$\begin{aligned}
& \frac{4}{9}N(N-1) \left[1 - \left(\frac{2}{3} \right)^{N-3} \right] - \frac{8}{9}(N-1)E_{N-2}(n_b + n_r + n_{br}) + \\
& + \frac{4}{9}E_{N-2}[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] \asymp \\
& \frac{4}{9}N(N-1) - \frac{8}{9}(N-1)E_{N-2}(n_b + n_r + n_{br}) + \\
& \frac{4}{9}E_{N-2}[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] \tag{3.20}
\end{aligned}$$

Putting together (3.18)-(3.20) we get (3.14) and Proposition 3.2 is so proved. \square

Now we are able to calculate the second factorial moment and the variance of $n_b + n_r$

Corollary 3.2 *For large N , the following asymptotics*

$$E_N[(n_b + n_r)(n_b + n_r - 1)] \asymp \frac{4}{81}(N-1)(N-3) \tag{3.21}$$

holds.

Proof: Since $n_b + n_r + n_{br} \sim B(N-1, \frac{1}{3})$ for sequences ξ_N of length N , one can easily find its second factorial moment

$$E_N[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] = \frac{(N-1)(N-2)}{9} \tag{3.22}$$

Analogously we have

$$\begin{aligned}
E_{N-1}[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] &= \frac{(N-2)(N-3)}{9} \\
E_{N-2}[(n_b + n_r + n_{br})(n_b + n_r + n_{br} - 1)] &= \frac{(N-3)(N-4)}{9} \tag{3.23}
\end{aligned}$$

From (3.5), (3.22), (3.23) and Proposition 3.2 we obtain the second factorial moment of $n_b + n_r$

$$\begin{aligned}
E_N[(n_b + n_r)(n_b + n_r - 1)] &\asymp \\
&\frac{N(N-1)}{9} - 2(N-1) \left[\frac{N-1}{3} - \frac{4}{3} \frac{N-2}{3} + \frac{4}{9} \frac{N-3}{3} \right] + \\
&\frac{(N-1)(N-2)}{9} - \frac{4}{3} \frac{(N-2)(N-3)}{9} + \frac{4}{9} \frac{(N-3)(N-4)}{9} = \\
&\frac{N(N-1)}{9} - \frac{2}{3}(N-1)^2 + (N-1)(N-2) - \frac{8}{27}(N-1)(N-3) + \\
&\quad - \frac{4}{27}(N-2)(N-3) + \frac{4}{81}(N-3)(N-4) = \\
&\frac{N-1}{9} [N + 9(N-2) - 6(N-1)] - \frac{32}{81}(N-1)(N-3) = \\
&\quad \frac{4}{81}(N-1)(N-3)
\end{aligned}$$

□

Corollary 3.3 *For N large enough*

$$Var_N(n_b + n_r) \asymp \frac{2}{81}(3N + 1) \quad (3.24)$$

The symbol Var_N indicates the variance associated to probability measure P_N .

Proof: One has that

$$\begin{aligned}
Var_N(n_b + n_r) &= E_N[(n_b + n_r)(n_b + n_r - 1)] + E_N[n_b + n_r] - (E_N[n_b + n_r])^2 \asymp \\
&\frac{4}{81}(N-1)(N-3) + \frac{2N-1}{9} - \frac{(2N-1)^2}{81} = \\
&\frac{1}{81}[4(N-1)(N-3) + 9(2N-1) - (2N-1)^2] = \\
&\quad \frac{2}{81}(3N + 1)
\end{aligned}$$

□

Remark 3.3 *Note that in the variance formula (3.24) for $n_b + n_r$ the second order term with respect to N disappears, so that the accuracy of the first order terms with respect to N is important. Nevertheless, from the asymptotics for the mean and the variance of $n_b + n_r$ ((3.6) and (3.24)) one can deduce that the distribution of $n_b + n_r$ is asymptotically not binomial. In fact $E_N(n_b + n_r) \asymp \frac{2}{9}N$ and $Var_N(n_b + n_r) \asymp \frac{2}{27}N \neq \frac{2}{9} \cdot \frac{7}{9}N$. In the next section we prove that it is asymptotically gaussian, for large N , i.e. a C.L.T. holds.*

4 Central Limit Theorem for dimers' number

In the present section we study the asymptotic distribution of the dimers' number, in particular we prove a C.L.T. for the joint probability distribution of the total number of dimers and single points $(n_b + n_r + \gamma_b + \gamma_r)$, analyzed in Section 2, and the number of dimers $(n_b + n_r)$. The limit distribution is a bivariate gaussian distribution with correlation coefficients equal to $-\frac{1}{\sqrt{3}}$. The proof is a generalization of De Moivre-Laplace's Theorem.

Theorem 4.1 *A C.L.T. holds for the joint probability distribution*

$$P_N(n_b + n_r = s; n_b + n_r + \gamma_b + \gamma_r = k) = \frac{1}{\sqrt{(2\pi)^2 \left(\frac{2}{27}N\right) \left(\frac{2}{9}N\right) \left(\frac{2}{3}\right)}} e^{-\frac{3}{4}\left(x^2 + \frac{2\sqrt{3}}{3}xy + y^2\right)} (1 + r_{x,y}(N)) \quad (4.1)$$

where

$$x = \frac{s - \frac{2}{9}N}{\frac{\sqrt{6N}}{9}} \quad y = \frac{k - \frac{2}{3}N}{\frac{\sqrt{2N}}{3}} \quad (4.2)$$

Moreover $\lim_{N \rightarrow \infty} r_{x,y}(N) = 0$, uniformly with respect to x and y , defined in (4.2), belonging to finite intervals $(-C, C)$ and $(-A, A)$ respectively, with C and A positive real constants.

Proof: As in Section 2 we perform the change of variables $k = N - t + s$ and $s = s$ on $\tilde{P}_N(s, t)$, defined in (2.6). The index k indicates the total number of dimers and single points. Then the probability $\tilde{P}_N(s, t)$ becomes

$$P_N(n_b + n_r = s; n_b + n_r + \gamma_b + \gamma_r = k) = \tilde{P}_N(s, N - k + s) = \binom{k}{s} \binom{N - k - 1}{s - 1} \left(\frac{2}{3}\right)^{k-1} \left(\frac{1}{3}\right)^{N-k} \quad (4.3)$$

Taking into account that the indices s and k are both of order N , we can forget in (4.3) the constants, i.e. $k - 1 \asymp k$ and $N - k - 1 \asymp N - k$, as $N \rightarrow \infty$.

As in De Moivre-Laplace's Theorem, we apply Stirling's formula to the binomial coefficients in (4.3). In the present model we have two binomial coefficients instead of one, so that the calculus becomes heavier than in De Moivre-Laplace's Theorem. We write

$$\binom{k}{s} \binom{N - k}{s} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{N-k} =$$

$$\begin{aligned}
& \frac{k!}{s!(k-s)!} \frac{(N-k)!}{s!(N-k-s)!} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{N-k} = \\
& \frac{k^k}{s^s(k-s)^{k-s}} \frac{(N-k)^{N-k}}{s^s(N-k-s)^{N-k-s}} \left(\frac{2}{3}\right)^k \left(\frac{1}{3}\right)^{N-k} \cdot \\
& \frac{e^{\lambda_{N,s,k}}}{\sqrt{2\pi \frac{s(k-s)}{k}} \sqrt{2\pi \frac{s(N-k-s)}{N-k}}} \quad (4.4)
\end{aligned}$$

where

$$\lambda_{N,s,k} \equiv \lambda_k - \lambda_s - \lambda_{k-s} + \lambda_{N-k} - \lambda_s - \lambda_{N-k-s}$$

and

$$\frac{1}{12n+1} \leq \lambda_n \leq \frac{1}{12n}, \text{ for any } n \in \mathbb{N}$$

Then we rewrite (4.4) as

$$\begin{aligned}
& \underbrace{e^{\lambda_{N,s,k}}}_{A_{N,s,k}} \cdot \frac{1}{\underbrace{\sqrt{2\pi \frac{s(k-s)}{k}} \sqrt{2\pi \frac{s(N-k-s)}{N-k}}}_{B_{N,s,k}}} \cdot \\
& \underbrace{\left(\frac{k}{3s}\right)^s \left(\frac{2k}{3(k-s)}\right)^{k-s} \left(\frac{2(N-k)}{3s}\right)^s \left(\frac{N-k}{3(N-k-s)}\right)^{N-k-s}}_{C_{N,s,k}} \quad (4.5)
\end{aligned}$$

In order to find an asymptotics for (4.5) we recall that the r.v. $n_b + n_r + \gamma_b + \gamma_r$ has binomial distribution $B(N-1, \frac{2}{3})$ (Corollary 2.1) and hence $E_N(n_b + n_r + \gamma_b + \gamma_r) \asymp \frac{2}{3}(N-1) \asymp \frac{2}{3}N$ and $Var_N(n_b + n_r + \gamma_b + \gamma_r) \asymp \frac{2}{9}(N-1) \asymp \frac{2}{9}N$. Moreover in the previous section we proved that $E_N(n_b + n_r) \asymp \frac{2N-1}{9} \asymp \frac{2}{9}N$ and that $Var_N(n_b + n_r) \asymp \frac{2}{81}(3N+1) \asymp \frac{6}{81}N$.

According to De Moivre-Laplace's Theorem we normalize the variables $n_b + n_r + \gamma_b + \gamma_r$ and $n_b + n_r$, by using the asymptotics of the moments and performing then the change of variables (4.2) so that

$$\begin{cases} s = \frac{2}{9}N + \frac{\sqrt{6N}}{9}x & \text{for } x \in (-C, C) \\ k = \frac{2}{3}N + \frac{\sqrt{2N}}{3}y & \text{for } y \in (-A, A) \end{cases} \quad (4.6)$$

Taking into account (4.6), we consider the first factor $A_{N,s,k}$ in (4.5) with respect to the variables x and y , in particular

$$\lambda_{N,s,k} \leq$$

$$\begin{aligned}
& \frac{1}{12 \left(\frac{2}{3}N + \frac{\sqrt{2N}}{3}y \right)} - \frac{2}{12 \left(\frac{2}{9}N + \frac{\sqrt{6N}}{9}x \right) + 1} - \frac{1}{12 \left(\frac{4}{9}N - \frac{\sqrt{6N}}{9}x + \frac{\sqrt{2N}}{3}y \right) + 1} \\
& + \frac{1}{12 \left(\frac{N}{3} - \frac{\sqrt{2N}}{3}y \right)} - \frac{1}{12 \left(\frac{N}{9} - \frac{\sqrt{6N}}{9}x - \frac{\sqrt{2N}}{3}y \right) + 1} \quad (4.7)
\end{aligned}$$

Since x, y belong to bounded intervals one can estimate $\lambda_{N,s,k}$ uniformly from above with respect to x and y and we get a bound of order $\frac{1}{N}$ that doesn't depend on x and y . Analogously, we can find a uniform lower bound of order $\frac{1}{N}$. Therefore

$$A_{N,s,k} = 1 + r_{x,y}^{(1)}(N)$$

with $r_{x,y}^{(1)}(N) \rightarrow 0$, as $N \rightarrow \infty$, uniformly with respect to x and y . The second factor in (4.5) is $B_{N,s,k}$

$$B_{N,s,k} = \frac{1}{\sqrt{2\pi \frac{s(k-s)}{k}} \sqrt{2\pi \frac{s(N-k-s)}{N-k}}}$$

From (4.6) we estimate

$$\begin{aligned}
\frac{s(k-s)}{k} &= \frac{\left(\frac{2}{9}N + \frac{\sqrt{6N}}{9}x \right) \left(\frac{4}{9}N - \frac{\sqrt{6N}}{9}x + \frac{\sqrt{2N}}{3}y \right)}{\left(\frac{2}{3}N + \frac{\sqrt{2N}}{3}y \right)} = \\
&= \frac{4}{27} N \frac{\left(1 + \sqrt{\frac{6}{N}} \frac{x}{2} \right) \left(1 - \sqrt{\frac{6}{N}} \frac{x}{4} + \frac{3}{4} \sqrt{\frac{2}{N}} y \right)}{\left(1 + \sqrt{\frac{2}{N}} \frac{y}{2} \right)} = \\
&= \frac{4}{27} N \left(1 + r_{x,y}^{(2)}(N) \right) \quad (4.8)
\end{aligned}$$

with $r_{x,y}^{(2)}(N) \rightarrow 0$, as $N \rightarrow \infty$, uniformly with respect to x and y , since $x \in (-C, C)$ and $y \in (-A, A)$, so that the upper and lower bounds of $B_{N,s,k}$ do not depend on x and y .

Analogously

$$\frac{s(N-k-s)}{N-k} = \frac{2}{27} N \left(1 + r_{x,y}^{(3)}(N) \right)$$

with $r_{x,y}^{(3)}(N) \rightarrow 0$, as $N \rightarrow \infty$, uniformly with respect to x and y . Therefore

$$B_{N,s,k} \asymp \frac{1}{2\pi \frac{\sqrt{6N}}{9} \frac{\sqrt{2N}}{3} \sqrt{\frac{2}{3}}} \quad (4.9)$$

as $N \rightarrow \infty$.

Note that

$$B_{N,s,k} \asymp \frac{1}{2\pi\sqrt{\text{Var}_N(n_b + n_r)}\sqrt{\text{Var}_N(n_b + n_r + \gamma_b + \gamma_r)}\sqrt{1 - \rho^2}}$$

where $\rho = \pm \frac{1}{\sqrt{3}}$ is the correlation coefficient, whose sign will be determined later.

Finally we consider the logarithm of the last factor $C_{N,s,k}$ in (4.5)

$$\begin{aligned} \ln C_{N,s,k} &= -s \ln \left(\frac{3s}{k} \right) - (k-s) \ln \left(\frac{3(k-s)}{2k} \right) + \\ &-s \ln \left(\frac{3s}{2(N-k)} \right) - (N-k-s) \ln \left(\frac{3(N-k-s)}{N-k} \right) \equiv \\ &\sum_{i=1}^4 C_{N,s,k}^i \end{aligned} \quad (4.10)$$

We express now each term of the sum in (4.10) $C_{N,s,k}^i$, $i = 1, 2, 3, 4$ in terms of x and y , defined in (4.2). We start with $C_{N,s,k}^1$

$$\begin{aligned} C_{N,s,k}^1 &\equiv -s \ln \left(\frac{3s}{k} \right) = \\ &- \left(\frac{2}{9}N + \frac{\sqrt{6N}}{9}x \right) \ln \left(\frac{\frac{2}{3}N + \frac{\sqrt{6N}}{3}x}{\frac{2}{3}N + \frac{\sqrt{2N}}{3}y} \right) = \\ &-\frac{\sqrt{2N}}{9} \left(\sqrt{2N} + \sqrt{3}x \right) \ln \left(1 + \frac{\sqrt{3}x - y}{\sqrt{2N} + y} \right) \end{aligned}$$

Since the last logarithm above is of the form $\ln(1+z)$, with $z \rightarrow 0$, we can expand it around $z = 0$, $\ln(1+z) = z - \frac{z^2}{2} + o(z^2)$, as $z \rightarrow 0$. The same is true for each logarithm function present in any $C_{N,s,k}^i$, $i = 1, 2, 3, 4$. So $C_{N,s,k}^1$ becomes, as $N \rightarrow \infty$,

$$C_{N,s,k}^1 \asymp -\frac{\sqrt{2N}}{18} \frac{(\sqrt{2N} + \sqrt{3}x)(\sqrt{3}x - y)(2\sqrt{2N} - \sqrt{3}x + 3y)}{(\sqrt{2N} + y)^2} \quad (4.11)$$

Analogously for $C_{N,s,k}^2$

$$C_{N,s,k}^2 \equiv -(k-s) \ln \left(\frac{3(k-s)}{2k} \right) =$$

$$\begin{aligned}
& - \left(\frac{4}{9}N - \frac{\sqrt{6N}}{9}x + \frac{\sqrt{2N}}{3}y \right) \ln \left(\frac{\frac{4}{3}N - \frac{\sqrt{6N}}{3}x + \sqrt{2N}y}{\frac{4}{3}N + 2\frac{\sqrt{2N}}{3}y} \right) = \\
& - \frac{\sqrt{2N}}{9} \left(2\sqrt{2N} - \sqrt{3}x + 3y \right) \ln \left(1 + \frac{-\sqrt{3}x + y}{2(\sqrt{2N} + y)} \right) \asymp \\
& + \frac{\sqrt{2N}}{72} \frac{\left(2\sqrt{2N} - \sqrt{3}x + 3y \right) (\sqrt{3}x - y) \left(4\sqrt{2N} + \sqrt{3}x + 3y \right)}{\left(\sqrt{2N} + y \right)^2} \quad (4.12)
\end{aligned}$$

Then $C_{N,s,k}^3$

$$\begin{aligned}
C_{N,s,k}^3 & \equiv -s \ln \left(\frac{3s}{2(N-k)} \right) = \\
& - \left(\frac{2}{9}N + \frac{\sqrt{6N}}{9}x \right) \ln \left(\frac{2N + \sqrt{6N}x}{2N - 2\sqrt{2N}y} \right) = \\
& - \frac{\sqrt{2N}}{9} \left(\sqrt{2N} + \sqrt{3}x \right) \ln \left(1 + \frac{\sqrt{3}x + 2y}{\sqrt{2N} - 2y} \right) \asymp \\
& - \frac{\sqrt{2N}}{18} \frac{\left(\sqrt{2N} + \sqrt{3}x \right) (\sqrt{3}x + 2y) \left(2\sqrt{2N} - \sqrt{3}x - 6y \right)}{\left(\sqrt{2N} - 2y \right)^2} \quad (4.13)
\end{aligned}$$

It remains to see $C_{N,s,k}^4$

$$\begin{aligned}
C_{N,s,k}^4 & \equiv -(N-k-s) \ln \left(\frac{3(N-k-s)}{N-k} \right) = \\
& - \left(\frac{N}{9} - \frac{\sqrt{6N}}{9}x - \frac{\sqrt{2N}}{3}y \right) \ln \left(\frac{\frac{N}{3} - \frac{\sqrt{6N}}{3}x - \sqrt{2N}y}{\frac{N}{3} - \frac{\sqrt{2N}}{3}y} \right) = \\
& - \frac{\sqrt{2N}}{9} \left(\sqrt{\frac{N}{2}} - \sqrt{3}x - 3y \right) \ln \left(1 - \frac{\sqrt{3}x + 2y}{\sqrt{\frac{N}{2}} - y} \right) \asymp \\
& 2\frac{\sqrt{2N}}{9} \frac{\left(\sqrt{\frac{N}{2}} - \sqrt{3}x - 3y \right) (\sqrt{3}x + 2y) \left(\sqrt{2N} + \sqrt{3}x \right)}{\left(\sqrt{2N} - 2y \right)^2} \quad (4.14)
\end{aligned}$$

Summing the last term in (4.11) ($C_{N,s,k}^1$) with that one in (4.12) ($C_{N,s,k}^2$) we obtain

$$C_{N,s,k}^1 + C_{N,s,k}^2 \asymp -\frac{\sqrt{2N}}{24} \frac{\left(2\sqrt{2N} - \sqrt{3}x + 3y\right)(\sqrt{3}x - y)^2}{\left(\sqrt{2N} + y\right)^2} \quad (4.15)$$

Summing the last term in (4.13) ($C_{N,s,k}^3$) with that one in (4.14) ($C_{N,s,k}^4$) we obtain

$$C_{N,s,k}^3 + C_{N,s,k}^4 \asymp -\frac{\sqrt{2N}}{6} \frac{\left(\sqrt{2N} + \sqrt{3}x\right)(\sqrt{3}x + 2y)^2}{\left(\sqrt{2N} - 2y\right)^2} \quad (4.16)$$

Finally, the main contribution of (4.15) and (4.16) is

$$\begin{aligned} \sum_{i=1}^4 C_{N,s,k}^i &\asymp -\frac{1}{12} \left[\left(\sqrt{3}x - y\right)^2 + 2\left(\sqrt{3}x + 2y\right)^2 \right] = \\ &= -\frac{3}{4} \left[x^2 + 2\frac{\sqrt{3}}{3}xy + y^2 \right] \end{aligned} \quad (4.17)$$

Remark 4.1 *Note that the last term in (4.17) is of the form*

$$-\frac{1}{2(1-\rho^2)} (x^2 - 2\rho xy + y^2)$$

with $\rho = -\frac{1}{\sqrt{3}}$. We get so a bivariate gaussian distribution with correlation coefficient equal to $-\frac{1}{\sqrt{3}}$, i.e., the r.v. $n_b + n_r$ and $n_b + n_r + \gamma_b + \gamma_r$ are negatively correlated.

From the previous theorem we can deduce the following result.

Corollary 4.1 *For large N , a C.L.T. for the r.v. $n_b + n_r$ holds, i.e.*

$$P(n_b + n_r = s) \asymp \frac{1}{\sqrt{2\pi \frac{2}{27}N}} e^{-\frac{\left(s - \frac{2}{9}N\right)^2}{\frac{4}{27}N}} \quad (4.18)$$

for any s defined in (4.6).

Proof: From (4.3) we have

$$P_N(n_b + n_r = s) = \sum_{k=s}^{N-s} \binom{k}{s} \binom{N-k-1}{s-1} \left(\frac{2}{3}\right)^{k-1} \left(\frac{1}{3}\right)^{N-k} =$$

$$\left(\sum_{k=\frac{2}{3}N+\frac{\sqrt{6N}}{9}x}^{\frac{2}{3}N-\frac{\sqrt{2N}}{3}A} + \sum_{k=\frac{2}{3}N-\frac{\sqrt{2N}}{3}A}^{\frac{2}{3}N+\frac{\sqrt{2N}}{3}A} + \sum_{k=\frac{2}{3}N+\frac{\sqrt{2N}}{3}A}^{\frac{7}{9}N-\frac{\sqrt{6N}}{9}x} \right) \tilde{P}_N(s, N-k+s) \quad (4.19)$$

In (4.19), let us denote by $\Sigma_1(A, N)$, $\Sigma_2(A, N)$ and $\Sigma_3(A, N)$ the first, second and third sum, respectively. Using Theorem 4.1 and the integral C.L.T. of De Moivre-Laplace [4] one finds for every $\epsilon > 0$ an $N(\epsilon)$ such that

$$|\Sigma_i(A, N) - I_i(A)| < \frac{\epsilon}{3}, \quad i = 1, 2, 3 \quad (4.20)$$

for any $N > N(\epsilon)$. The terms $I_i(A)$ are given by

$$\frac{1}{\sqrt{2\pi \frac{2}{27}N}} e^{-\frac{x^2}{2}} \frac{1}{\sqrt{2\pi}} \int_i e^{-\frac{y^2}{2}} dy$$

and the boundaries of the integrals \int_i are fixed as $\int_1 = \int_{-\infty}^{-A}$, $\int_2 = \int_{-A}^A$ and $\int_3 = \int_A^{\infty}$, respectively. Note that (4.20) proves the statement. \square

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